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Vibration and buckling of rotating flexible rods at transitional parameter values

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SUMMARY

We consider the connection between vibration and buckling problems for a uniform flexible rod which is clamped at one end and rotates in a plane perpendicular to the axis of rotation. The rod is assumed off-clamped, i.e. the axis of rotation does not pass through the rod's clamped end. The resulting fourth-order boundary value problem with a turning point for the free vibrations is solved using uniform approximations in a transitional parameter range where high rotation rates balance small off-clampings. Second approximations to the vibration eigenvalues are used to determine critical buckling rotation rates for the slightly off-clamped rods. Buckling is unexpected in this situation as the rod is wholly under tension.

1. Introduction

In this work, we consider several boundary value problems which arise in connection with the vibration and buckling of a uniform flexible rod which is clamped at one end and rotates in a plane perpendicular to the axis of rotation. The axis of rotation, however, is not assumed to pass through the rod's clamped end. Rather, as the rod rotates, its clamped end describes a circle of radius $R > 0$ about the axis of rotation. If the rod has length L, then the degree of off-clamping is described by the dimensionless parameter $\alpha = R/L$. Small α, i.e. $R \ll L$, corresponds to a wobbling hub-clamped rotor, while $\alpha > 1$ corresponds to a rod which is clamped to the rim of a rotating wheel and extends inward toward the center like a partial spoke. Recent interest in rotating rods stems from applications to the dynamic stability of satellite antennas, helicopter rotor blades, turbine blades, and energy-storing flywheels. Rotation rates in these applications are usually high.

Let the rod have cross-sectional area A, mass per unit volume ρ, bending stiffness EI, and constant angular velocity Ω. We will assume that the rod does not twist, and for the vibration problem we seek displacements with harmonic time dependence $e^{i\omega t}$. If the amplitude of the vibrations is sufficiently small that we can consistently linearize, then the equation governing the free vibrations in a plane making an angle θ with the plane of rotation $(0 \leq \theta \leq \pi/2)$ is

$$\tilde{\epsilon}^3 w^{iv} - \tfrac{1}{2}(1+x)(1-2\alpha-x)w''$$
$$+ (x+\alpha)w' - (\lambda + \cos^2\theta)w = 0 \tag{1.1}$$

where

$$\tilde{\epsilon}^3 = EI/\rho A \Omega^2 L^4 \quad \text{and} \quad \lambda = (\omega/\Omega)^2. \tag{1.2}$$

For rapid rotation rates, $\tilde{\epsilon}$ is small, real, and positive. The zero displacement and slope requirements at the clamped end $x = 0$ and the vanishing of stress and bending moment at the free end $x = -1$ lead to the associated boundary conditions

$$w(0) = w'(0) = w''(-1) = w'''(-1) = 0. \tag{1.3}$$

Positive eigenvalues $\lambda(\alpha, \tilde{\epsilon}, \theta)$ of the boundary value problem (1.1) and (1.3) now give the rotating rod's natural frequencies of vibration $\omega = \lambda^{1/2}\Omega$.

Time-independent deformed states (buckled modes) of the rotating rod are governed by the equation

$$u^{iv} - \mu^2\{\tfrac{1}{2}(1+x)(1-2\alpha-x)u'' - (x+\alpha)u' + (\cos^2\theta)u\} = 0 \tag{1.4}$$

where $\mu^2 = \tilde{\epsilon}^{-3}$ and θ is the angle between the plane of rotation and the plane in which buckling occurs. Associated boundary conditions are again (1.3), and the positive eigenvalues $\mu\,(\alpha,\theta)$ of the boundary value problem (1.3) and (1.4) determine the critical rotation rates $\Omega = \mu[EI/\rho AL^4]^{1/2}$ for buckling.

The use of vibration results to predict buckling for rapid rotation rates has been explored previously by Lakin and Nachman [4] in the off-clamped rod context. For fixed small $\tilde{\epsilon}$ and α in the range $O(\epsilon^{3/2}) < \alpha < 1/2$, it was found that there is a critical angle θ_c for which the lowest vibration eigenvalue λ_0 vanishes. Comparing (1.1) and (1.4), the corresponding 'vibration' eigenfunction is now a solution of the buckling boundary value problem (1.3) and (1.4) with positive μ^2 and hence represents a time-independent ($\omega_0 = \lambda_0^{1/2}\Omega = 0$) buckled mode in the θ_c-plane. Indeed, the curve $\theta = \theta_c(\alpha)$, for which $\lambda_0 = 0$, is a buckling boundary. For $\theta < \theta_c$, Lakin and Nachman [4] also found that the lowest vibration eigenvalue λ_0 is negative. One of the square roots of λ_0 is thus purely imaginary and negative so $e^{i\omega_0 t} = e^{+\sigma_0 t}$ where $\sigma_0 = \Omega|\lambda_0|^{1/2} > 0$. Hence, in planes with $\theta < \theta_c$ small amplitude disturbances of the rod will grow without oscillation and diverge. In a fluid mechanics context, this behavior would be termed an instability.

As $\cos\theta$ is a decreasing function of θ for $0 \leq \theta \leq \pi/2$, for given α and $\tilde{\epsilon}$, λ_0 will be an increasing function of θ. Hence, buckling or divergence will first be observed in the plane of rotation where $\theta = 0$. For physical applications, in-plane buckling, and hence in-plane vibrations, will thus be most important. In treating the vibration equation (1.1), however, it is considerably more convenient to obtain the eigenvalues for transverse vibrations $\theta = \pi/2$. In-plane eigenvalues may then be obtained from the transverse eigenvalues through the simple relation

$$\lambda(\alpha, \tilde{\epsilon}, 0) = \lambda(\alpha, \tilde{\epsilon}, \pi/2) - 1. \tag{1.5}$$

Results for transverse vibration obtained by Lakin [2], coupled with (1.5), show that for $\alpha < O(\tilde{\epsilon}^{3/2})$, $\lambda_0(\alpha, \tilde{\epsilon}, 0)$ is positive, so the rotating rod can sustain small vibrations and does not buckle. For $\alpha > O(\epsilon^{1/2})$, Lakin and Nachman [4] found that while λ_0 may be zero or negative, the higher vibration eigenvalues $\lambda_n(\alpha, \tilde{\epsilon}, \theta)$ were positive for $\alpha < 1/2$, $0 \leq \theta \leq \pi/2$, and $n \geq 1$, so the rod has exactly one buckled mode.

In this paper, we concentrate on the transition case (the distinguished limit) where slight off-clamping and rapid rotation combine to give $\alpha = O(\epsilon^{3/2})$. A principal objective is to determine the constant k such that when $\alpha \geq k\ \epsilon^{3/2}$ in-plane buckling is indicated. As α is small and $\mu \sim 1/\alpha$, the in-plane buckled mode in this transition range will correspond to a large buckling eigenvalue, i.e. a high rotation rate.

2. Vibration eigenvalue for $\alpha = O(\tilde{\epsilon}^{3/2})$

As $\tilde{\epsilon}$ is small for rapid rotation rates, asymptotic methods may be used to approximate solutions of equation (1.1). In particular, the reduced equation obtained by setting $\tilde{\epsilon}$ to zero is only of second order, so singular perturbation techniques are required. For $\alpha < 1/2$, the distinctive character of the boundary value problem now comes from the simple turning point at the endpoint $x = -1$ where the coefficient of w'' in (1.1) vanishes. A further complication here is that one solution of the reduced equation has a logarithmic singularity at the turning point, whereas the corresponding solution of the full equation is regular at the turning point.

The off-clamping paramter α may be removed from equation (1.1) by defining new independent and dependent variables

$$y = \frac{x + \alpha}{1 - \alpha} \quad \text{and} \quad \phi(y) = w(x). \tag{2.1}$$

For $\theta = \pi/2$, the equation (1.1) for transverse vibrations becomes

$$\epsilon^3 \phi^{iv} - \tfrac{1}{2}(1 - y^2)\phi'' + y\phi' - \lambda\phi = 0 \tag{2.2}$$

where $\epsilon^3 = \tilde{\epsilon}^3(1 - \alpha)^{-4}$. The parameter α now appears in the transformed boundary conditions

$$\phi\left(\frac{\alpha}{1 - \alpha}\right) = \phi'\left(\frac{\alpha}{1 - \alpha}\right) = \phi''(-1) = \phi'''(-1) = 0. \tag{2.3}$$

A set of four linearly independent *exact* solutions of equation (2.2) which are 'numerically satisfactory' in the sense of Miller [6] may be defined to within multiplicative constants by their asymptotic properties away from the turning point in $[-1, \alpha/(1 - \alpha)]$. The solution U_0 is well balanced, U_1 is balanced but logarithmic, V_1 is recessive, while V_2 is dominant for $-1 < y \leq \alpha/(1 - \alpha)$. Care must be taken with U_1 as adding multiples of U_0 will not effect the defining asymptotic property.

Approximations to solutions of equation (2.2) have been derived by Lakin [1], Lakin and Ng [5], and Peters [7]. In the present work, however, we will use uniformly valid approximations obtained by Lakin [3] and Lakin and Nachman [4]. These approximations involve Airy and Scorer functions, and the Langer variable

$$\eta = \tfrac{1}{2}\{3 \int_{-1}^{y} (1 - y^2)^{1/2} dy\}^{2/3} \tag{2.4}$$

which explicity brings out the turning point nature of equation (2.1). In particular, if $\zeta = \eta/\epsilon$ is a stretched Langer variable, then uniform approximations are

$$U_0(\eta) \simeq \overline{\phi}_1^{(0)}(y) + O(\epsilon^3), \tag{2.5}$$

$$U_1(\eta) = H(\eta) + a(\eta)\,Gi(\zeta,1) + \epsilon^2 b(\eta)\,Gi(\zeta,0) + \epsilon c(\eta)\,Gi(\zeta,-1) + O(\epsilon^3), \tag{2.6}$$

$$V_1(\eta) = a(\eta)\,Ai(\zeta,1) + \epsilon^2 b(\eta)\,Ai(\zeta,0) + \epsilon c(\eta)\,Ai(\zeta,-1) + O(\epsilon^3) \tag{2.7}$$

and

$$V_2(\eta) = a(\eta)\,Bi(\zeta,1) + \epsilon^2 b(\eta)\,Bi(\zeta,0) + \epsilon c(\eta)\,Bi(\zeta,-1) + O(\epsilon^3) \tag{2.8}$$

where

$$Ai(\zeta,1) = \int_0^\zeta Ai(t)dt - 1/3, \quad Bi(\zeta,1) = \int_0^\zeta Bi(t)dt,$$
$$Gi(\zeta,1) = \pi \int_0^\zeta Gi(t)dt,$$

$$Ai(\zeta,0) = Ai(\zeta), \quad Bi(\zeta,0) = Bi(\zeta),$$
$$Gi(\zeta,0) = \pi Gi(\zeta),$$

$$Ai(\zeta,-1) = Ai'(\zeta), \quad Bi(\zeta,-1) = Bi'(\zeta),$$
$$Gi(\zeta,-1) = \pi Gi'(\zeta),$$

Ai and Bi are the usual Airy functions and Gi is the Scorer function. In (2.5), $\overline{\phi}_1^{(0)}(y)$ is the regular solution of the reduced equation $\frac{1}{2}(1-y^2)\phi'' - y\phi' + \lambda\phi = 0$, and if $\nu(\nu+1) = 2\lambda$ then

$$\overline{\phi}_1^{(0)}(y) = P_\nu(-y) \tag{2.9}$$

where P_ν is the Legendre function of degree ν. Coefficients in these expansions are given by

$$\begin{aligned} a(\eta) &= \overline{\phi}_1^{(0)}(y), \\ c(\eta) &= \eta^{-1}\{\eta'^{-3/2} - a(\eta)\}, \\ b(\eta) &= \eta^{-1}c(\eta) + \eta'^{-3/2}\eta^{-1/2}\{G_1(\eta) - \frac{41}{48}\eta^{-3/2}\}, \end{aligned} \tag{2.10}$$

and

$$H(\eta) = R(y) + \{\log\epsilon + \tfrac{1}{3}[2\psi(1) - \log 3]\}a(\eta)$$

where $\psi(x)$ is the digamma function, $R(y)$ is the regular portion of the singular solution of the reduced equation, and $G_1(\eta)$ is the order $\epsilon^{3/2}$ term in the WKB approximation to the solutions V_1 and V_2.

If $X(\eta)$ denotes a general solution of equation (2.2), then in terms of η the boundary conditions (2.3) are

$$X(\eta_1) = X'(\eta_1) = 0 \tag{2.11}$$

where $\eta_1 = \eta\left(\dfrac{\alpha}{1-\alpha}\right)$, while at the turning point $\eta(-1) = 0$

$$B_2 X(0) = B_3 X(0) = 0, \tag{2.12}$$

with

$$B_2 = \frac{d^2}{d\eta^2} + \gamma(\eta)\frac{d}{d\eta}, \quad B_3 = \frac{d^3}{d\eta^3} + (\gamma'(\eta) - \gamma^2(\eta))\frac{d}{d\eta} \tag{2.13}$$

and $\gamma(\eta) = \eta''(y)/[\eta'(y)]^2$. Writing X as a linear combination of U_0, U_1, V_1, and V_2 and applying (2.11) and (2.12) now gives a characteristic equation for λ of the form

$$\Delta(\alpha, \lambda, \epsilon) = 0 \tag{2.14}$$

where Δ involves a four-by-four determinant. Fortunately, considerable initial simplification is possible. As the boundary point η_1 is well away from the turning point $\eta = 0$, with exponentially small error only dominant terms in Δ need be retained. Further $d/d\eta = \epsilon^{-1} d/d\zeta$ and approximations to $U_0(\eta)$ do not involve the stretched variable ζ. This implies that if $W(X, Y)$ is the Wronskian of $X(\eta)$ and $Y(\eta)$ evaluated at η_1 while $B(X, Y) = B_3 X(0) B_2 Y(0) - B_2 X(0) B_3 Y(0)$, then

$$\Delta = B(U_1, V_1)\, W(U_0, V_2)\{1 + O(\epsilon^3)\}. \tag{2.15}$$

Great care must be taken when approximations to solutions of equation (2.2) are used to approximate expressions like (2.14). In the present case, for example, the uniform approximations to V_1 and U_1 involve rapidly varying Airy and Scorer functions of a stretched variable. Hence, the term $B(U_1, V_1)$ in (2.15) involves differences of products of rapidly varying functions. Indeed, when the uniform approximations (2.6) and (2.7) are used in $B(U_1, V_1)$, a typical term which results is the Wronskian of the Scorer function Gi with the Airy function Ai. Fortunately, potential difficulties can be avoided through the use of relations like

$$Gi(x, 0)\, Ai(x, -1) - Gi(x, -1)\, Ai(x, 0) = -Ai(x, 1) \tag{2.16}$$

which reduce differences of products of rapidly varying functions to a single rapidly varying function. We now obtain the uniform approximation

$$\begin{aligned} 3^{1/3}\epsilon^5\Gamma(1/3) B(U_1, V_1) = 1 &- \epsilon\Gamma(1/3)(\lambda - 1/10)/3^{1/3}\Gamma(2/3) \\ &- \epsilon^2 3^{-2/3}\Gamma(1/3)\left\{\frac{5}{6}\lambda^2 - \frac{8}{15}\lambda + \frac{59}{350}\right\} \\ &+ O(\epsilon^3) \end{aligned} \tag{2.17}$$

where $\Gamma(x)$ is the gamma function. Similarly,

$$\begin{aligned}\epsilon W(U_0, V_2) = {} & \eta_1'^{-3/2}\overline{\phi}_1^{(0)}(y_1)\, Bi(\zeta_1) \\ & + \epsilon^2\{(b + c')(\eta_1)\,\overline{\phi}_1^{(0)}(y_1) \\ & - c(\eta_1)\eta_1'^{-1}\overline{\phi}_1^{(0)\prime}(y_1)\}Bi'(\zeta_1) + O(\epsilon^3)\end{aligned} \tag{2.18}$$

where $\eta_1' = \eta'(y_1), y_1 = \alpha/(1-\alpha)$, and $\zeta_1 = \eta_1/\epsilon$.

Equations (2.17) and (2.18) imply that for $\alpha = O(\epsilon^{3/2})$, the vibration eigenvalue λ should be expanded in powers of either α or $\epsilon^{3/2}$. We therefore take

$$\lambda = \lambda^{(0)} + \epsilon^{3/2}\lambda^{(1)} + O(\epsilon^3). \tag{2.19}$$

As $\nu(\nu+1) = 2\lambda$, we also expand ν as

$$\nu = \nu^{(0)} + \epsilon^{3/2}\nu^{(1)} + O(\epsilon^3)\,, \tag{2.20}$$

so that

$$\lambda^{(0)} = \tfrac{1}{2}\nu^{(0)}(\nu^{(0)} + 1) \quad \text{and} \quad \lambda^{(1)} = (\nu^{(0)} + \tfrac{1}{2})\nu^{(1)}. \tag{2.21}$$

Using (2.17) through (2.20) in (2.15), setting the resulting approximation to $\Delta(\alpha, \lambda, \epsilon)$ to zero, and expanding quantities evaluated at y_1 in MacLaurin series now shows that first approximations to the eigenvalues come from the relation

$$\overline{\phi}_1^{(0)}(y_1) = 0$$

which by (2.9) gives $\nu_n^{(0)}$ is an odd positive integer,

$$\nu_n^{(0)} = 2n + 1 \quad (n = 0, 1, 2, \ldots). \tag{2.22}$$

Second approximations to ν_n are found to be

$$\nu_n^{(1)} = \frac{4\Gamma\left(\frac{2n+3}{2}\right)}{\pi\, n!}\{2^{1/2} - \alpha\epsilon^{-3/2}\}. \tag{2.23}$$

Thus, to order ϵ^3, the lowest eigenvalue λ_0 for transverse vibrations $\left(\theta = \frac{\pi}{2}\right)$ is

$$\lambda_0(\alpha, \epsilon, \pi/2) = 1 + 3\pi^{-1/2}(2^{1/2}\epsilon^{3/2} - \alpha) + O(\epsilon^3). \tag{2.24}$$

The lowest vibration eigenvalue for in-plane vibrations ($\theta = 0$) may be obtained from (2.24) by noting that $\tilde{\epsilon} = \epsilon(1 + O(\alpha))$ and using (1.5). This gives

$$\lambda_0(\alpha, \tilde{\epsilon}, 0) = 3\pi^{-1/2}(2^{1/2}\epsilon^{3/2} - \alpha) + O(\epsilon^3). \tag{2.25}$$

For $\alpha = O(\epsilon^{3/2})$, as $\lambda_n^{(0)}(\alpha, \tilde{\epsilon}, 0) \simeq n(2n + 3)$, the higher eigenvalues $\lambda_n(\alpha, \tilde{\epsilon}, 0)$ with $n \geq 1$ are always positive.

3. In-plane buckling for slight off-clamping

Buckling is unexpected in the transition range of slight off-clamping and high rotation rates (i.e., $\alpha = O(\epsilon^{3/2})$) as each point of the rod is under tension. However, results for the vibration eigenvalues predict the possibility of exactly one in-plane buckled mode. By relation (2.25), the lowest in-plane vibration eigenvalue vanishes to order ϵ^3 when

$$\alpha = 2^{1/2}\epsilon^{3/2} \tag{3.1}$$

and is negative for $\alpha > 2^{1/2}\epsilon^{3/2}$. Higher in-plane vibration eigenvalues are always positive. Hence, as $\mu^2 = \tilde{\epsilon}^{-3}$, for $\alpha = 2^{1/2}\epsilon^{3/2}$ the in-plane vibration eigenfunction corresponding to λ_0 will be an in-plane buckled mode with eigenvalue

$$\mu^2 = \frac{2}{\alpha^2}\{1 + O(\alpha)\} \tag{3.2}$$

In particular, the transitional parameter range yields large buckling eigenvalues. By (3.1), critical rotation rates Ω_c for buckling at slight off-clampings are proportional to α^{-1}. Hence, the critical rotation rates for in-plane buckling are high and increase as the degree of off-clamping is reduced. In particular, buckling of axis-clamped rods ($\alpha \equiv 0$) is not indicated as infinite rotation rates would be required.

We note that to obtain (3.1) we have not had to solve the buckling boundary value problem (1.3) and (1.4) directly. Rather, we have exploited the connection between the vibration and buckling problems. The buckling problem is often difficult to solve directly as it contains one less parameter than the corresponding vibration problem. Hence, the ability to predict buckling behavior from vibration eigenvalues is highly useful.

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